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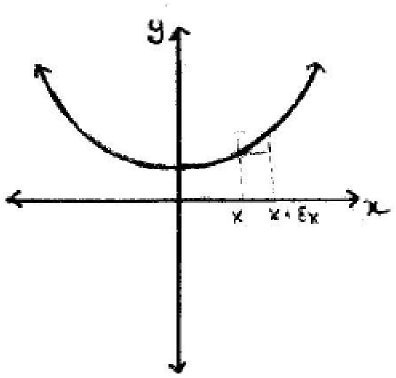
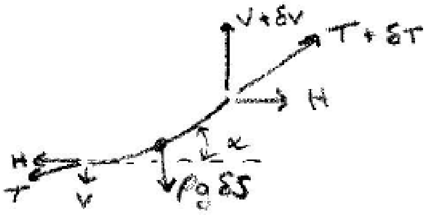
A	30/09/2017	JRB							

Derive catenary cable from first principles (1/3)





$$\sum F_V = 0$$

$$V(x + \delta x) = V(x) + \rho g \, \delta s$$

$$V(x + \delta x) - V(x) = \rho g \, \delta s$$

Take limit as $\delta x \to 0$ & divide by dx

$$\frac{dV}{dx} = \rho g \frac{ds}{dx} \quad — \text{①}$$

From Pythagoras

$$ds = \sqrt{dx^2 + dy^2}$$

dividing by dx:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad — \text{②}$$

Subbing ② into ①

$$\frac{dV}{dx} = \rho g \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad — \text{③}$$

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Derive catenary cable from first principles (2/3)

Evaluating Force Components @ x

$$V = T \sin\alpha$$

$$H = T \cos\alpha$$

$$\Rightarrow \frac{V}{H} = \tan\alpha = \frac{dy}{dx} \quad — (4)$$

So, $V = H \frac{dy}{dx}$

$$\therefore \frac{dV}{dx} = H \frac{d^2y}{dx^2} \quad — (5)$$

From (3) & (5)

$$H \frac{d^2y}{dx^2} = \rho g \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Let $\frac{H}{\rho g} = a$

$$\Rightarrow a y'' = \sqrt{1 + (y')^2}$$

Let $u = \frac{dy}{dx}$ to reduce to first order ODE

$$a u' = \sqrt{1 + u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1+u^2}} = \frac{1}{a} dx$$

$$\text{So, } \int \frac{du}{\sqrt{1+u^2}} = \frac{1}{a} \int dx \quad — (6)$$

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Derive catenary cable from first principles (3/3)

Using Identity $\cosh^2\theta - \sinh^2\theta = 1$

$$\cosh^2\theta = 1 + \sinh^2\theta$$

$$\Rightarrow \cosh\theta = \sqrt{1+\sinh^2\theta}$$

So, let $u = \sinh\theta$

$$\therefore \frac{du}{d\theta} = \cosh\theta$$

Now, (6) becomes:

$$\int d\theta = \frac{1}{a}\int dx$$

$$\Rightarrow \theta = \frac{x}{a} + c$$

Substituting back in for u

$$u = \sinh\left(\frac{x}{a}+c\right) = \frac{dy}{dx} \quad — (7)$$

$$y = \int \sinh\left(\frac{x}{a}+c\right) dx$$

$$\Rightarrow y = a\cosh\left(\frac{x+c}{a}\right) + k \quad — (8)$$

General form where C + k are arbitrary constants.

Consider arrangement drawn on page 1:

$\frac{dy}{dx} = 0$ at $x = 0$

$\therefore \sinh(c) = 0$

$\Rightarrow c = 0$

$y = a + k$ at $x = 0$

So if y intercept is 'a'

$\Rightarrow k = 0$

So the equation for the catenary curve in its simplest form is:

$$\boxed{y = a\cosh\left(\frac{x}{a}\right)} \quad — (9)$$

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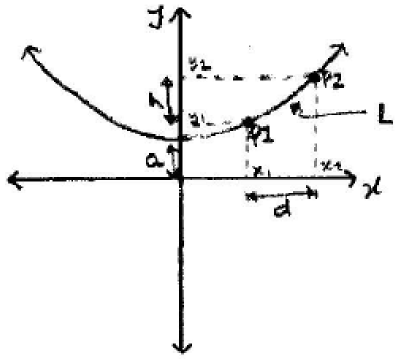
Fit catenary curve between 2 points (1/3)



$$y = a\cosh\left(\frac{x}{a}\right) \quad — \quad (1)$$

$$d = x_2 - x_1$$

$$h = a\cosh\left(\frac{x_2}{a}\right) - a\cosh\left(\frac{x_1}{a}\right)$$

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\, dx$$

$$= \int_{x_1}^{x_2} \cosh\left(\frac{x}{a}\right) dx$$

$$= a\sinh\left(\frac{x}{a}\right)\Big|_{x_1}^{x_2}$$

$$\Rightarrow \quad L = a\sinh\left(\frac{x_2}{a}\right) - a\sinh\left(\frac{x_1}{a}\right)$$

USING THE IDENTITIES: (theasuras.math.org)

$$\cosh(x) - \cosh(y) = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

$$\sinh(x) - \sinh(y) = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \quad h = 2a\sinh\left(\frac{x_2+x_1}{2a}\right)\sinh\left(\frac{x_2-x_1}{2a}\right) \quad — \quad (3)$$

$$\Rightarrow \quad L = 2a\cosh\left(\frac{x_2+x_1}{2a}\right)\sinh\left(\frac{x_2-x_1}{2a}\right) \quad — \quad (4)$$

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Fit catenary curve between 2 points (2/3)

NEED TO SIMPLIFY ③ + ④ TO SOLVE FOR "a"

$$L^2 - h^2 = 4a^2\left[\cosh^2\left(\frac{x_2+x_1}{2a}\right)\sinh^2\left(\frac{x_2-x_1}{2a}\right) - \sinh^2\left(\frac{x_2+x_1}{2a}\right)\sinh^2\left(\frac{x_2-x_1}{2a}\right)\right]$$

$$= 4a^2\sinh^2\left(\frac{x_2-x_1}{2a}\right)\left[\cosh^2\left(\frac{x_2+x_1}{2a}\right) - \sinh^2\left(\frac{x_2+x_1}{2a}\right)\right]$$

Using $\cosh^2\theta - \sinh^2\theta = 1$

$$\Rightarrow L^2 - h^2 = 4a^2\sinh^2\left(\frac{x_2-x_1}{2a}\right)$$

$$\therefore 2a\sinh\left(\frac{d}{2a}\right) = \sqrt{L^2-h^2} \quad — ⑤$$ where $d = x_2 - x_1$

CAN NOW SOLVE FOR "a" NUMERICALLY

Then, dividing ③ by ④

$$\frac{h}{L} = \tanh\left(\frac{x_1+x_2}{2a}\right)$$

$$\Rightarrow x_1 + x_2 = 2a\tanh^{-1}\left(\frac{h}{L}\right) \quad — ⑥$$

Using $\operatorname{arctanh} x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$

$$x_1 + x_2 = a\ln\left(\frac{1+\frac{h}{L}}{1-\frac{h}{L}}\right)$$

Times inside brackets by $\frac{L}{L}$

$$x_1 + x_2 = a\ln\left(\frac{L+h}{L-h}\right) \quad — ⑦$$

$$x_1 + (d + x_1) = a\ln\left(\frac{L+h}{L-h}\right)$$

$$\Rightarrow x_1 = \frac{1}{2}\left[a\ln\left(\frac{L+h}{L-h}\right) - d\right] \quad — ⑧$$

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Fit catenary curve between 2 points (3/3)

So, the objective is to find the equation to the catenary profile for a cable hanging between 2 points (P_{1x}, P_{1y}) (P_{2x}, P_{2y}).

Where $y = a \cosh\left(\frac{x+k}{a}\right) + c$

By solving for x_1 using (8) the remaining parameters can be found

$$k = x_1 - P_{1x} \quad - (9)$$

$$c = P_{1y} - a \cosh\left(\frac{x_1}{a}\right) \quad - (10)$$

Proposed method to solve for "a"

$$g(a) = 2a \sinh\left(\frac{d}{2a}\right) - \sqrt{L^2 - h^2} \quad - (1)$$

$$g'(a) = \left(\frac{d}{da} 2a\right) \sinh\left(\frac{d}{2a}\right) + 2a\left(\frac{d}{da} \sinh\left(\frac{d}{2a}\right)\right)$$

$$g'(a) = 2 \sinh\left(\frac{d}{2a}\right) + (2a)\cdot\left(-\frac{d}{2a^2}\right)\cdot \cosh\left(\frac{d}{2a}\right)$$

$$g'(a) = 2 \sinh\left(\frac{d}{2a}\right) - \frac{d}{a} \cosh\left(\frac{d}{2a}\right) \quad - (2)$$

Using NEWTON-RAPHSON METHOD

$$a_1 = a_0 - \frac{g(a_0)}{g'(a_0)} \quad - (3)$$

Where initial guess $a_0 > 0$.

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Spreadsheet (1/2)

JRB Catenary Cable V20170930 — JRB STRUCTURAL ENGINEERING PTY LTD

	x	y
Left	0	0
Right	20	5

L =	23	m
ρ =	10	kg/m
d =	20	m
h =	5	m
H =	1164.731	N

x_cat =	10.31523	m
x_lin =	9.237598	m
y_cat =	-2.00111	m
y_lin =	2.3094	m
max_sag =	4.443174	m

Solving Striaght Line	
m1 =	0.25
c1 =	0
Normal Line	
m2 =	-4

Solving Catenary Profile	
Solving for a	
a =	11.873
Error =	2.07E-08
Iterations =	16
Exit Flag =	1
Now Solving Parameters	
x1 =	-7.377
k =	-7.377
w =	-14.239

x_lowest =	7.38	m
y_lowest =	-2.37	m

y = a*cosh((x+k)/a)+w
y = 11.87*cosh((x-7.38)/11.87)-14.24

y' = sinh((x+k)/a)
y' = sinh((x-7.38)/11.87)

Li (m)	x (m)	y (m)	y'	V (N)	T (N)	xint (m)	yint (m)	sag (m)
0	0	0	-0.6621	-771.1637	1396.8868	0.0000	0.0000	0.0000
0.23	0.192629	-0.12567	-0.6427	-748.6007	1384.5585	0.1517	0.0379	0.1686
0.46	0.386963	-0.24869	-0.6234	-726.0377	1372.4905	0.3057	0.0764	0.3351
0.69	0.582995	-0.36898	-0.6040	-703.4747	1360.6895	0.4619	0.1155	0.4994
0.92	0.780714	-0.48648	-0.5846	-680.9117	1349.1627	0.6203	0.1551	0.6613
1.15	0.980108	-0.60112	-0.5652	-658.3487	1337.9172	0.7810	0.1953	0.8209
1.38	1.181164	-0.71281	-0.5459	-635.7857	1326.9600	0.9440	0.2360	0.9780
1.61	1.383864	-0.82149	-0.5265	-613.2227	1316.2984	1.1092	0.2773	1.1326
1.84	1.58819	-0.92709	-0.5071	-590.6597	1305.9396	1.2766	0.3192	1.2846
2.07	1.794118	-1.02952	-0.4877	-568.0967	1295.8909	1.4463	0.3616	1.4339
20.93	18.66638	3.41744	1.1007	1282.0693	1732.1377	18.3725	4.5931	1.2119
21.16	18.8203	3.588349	1.1201	1304.6323	1748.9038	18.5575	4.6394	1.0834
21.39	18.97274	3.760574	1.1395	1327.1953	1765.7991	18.7415	4.6854	0.9533
21.62	19.12372	3.934078	1.1589	1349.7583	1782.8199	18.9245	4.7311	0.8216
21.85	19.27327	4.108824	1.1782	1372.3213	1799.9625	19.1063	4.7766	0.6883
22.08	19.42139	4.284778	1.1976	1394.8843	1817.2235	19.2871	4.8218	0.5535
22.31	19.56811	4.461905	1.2170	1417.4473	1834.5997	19.4669	4.8667	0.4173
22.54	19.71343	4.640172	1.2363	1440.0103	1852.0877	19.6456	4.9114	0.2796
22.77	19.85739	4.819547	1.2557	1462.5733	1869.6844	19.8233	4.9558	0.1405
23	20	5	1.2751	1485.1363	1887.3868	20.0000	5.0000	0.0000

Max Design Tension = 1.89 kN (unfactored)

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Spreadsheet (2/2)